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Jacobi elliptic function solutions of nonlinear wave equations via the new sinh-Gordon equation expansion method

Zhenya Yan

Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

E-mail: zhenya_yan@yahoo.com

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Abstract

In this paper, based on the well-known sinh-Gordon equation, a new sinh-Gordon equation expansion method is developed. This method transforms the problem of solving nonlinear partial differential equations into the problem of solving the corresponding systems of algebraic equations. With the aid of symbolic computation, the procedure can be carried out by computer. Many nonlinear wave equations in mathematical physics are chosen to illustrate the method such as the KdV-mKdV equation, (2+1)-dimensional coupled Davey–Stewartson equation, the new integrable Davey–Stewartson-type equation, the modified Boussinesq equation, (2+1)-dimensional mKP equation and (2+1)-dimensional generalized KdV equation. As a consequence, many new doubly-periodic (Jacobian elliptic function) solutions are obtained. When the modulus $m \to 1$ or 0, the corresponding solitary wave solutions and singly-periodic solutions are also found. This approach can also be applied to solve other nonlinear differential equations.

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1. Introduction

Up to now more and more nonlinear evolution equations were presented which described the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamic, plasma physics, nonlinear optic, etc. The investigation of exact solutions of these nonlinear evolution equations is interesting and important. In the past few decades, many authors had mainly studied soliton solutions of nonlinear wave equations by using various methods, such as Backlund transformation [1, 5], Darboux transformation [2], inverse scattering method [3], Hirota's bilinear method [4], the tanh method [6], the sine–cosine method [7, 10], the homogeneous balance method [8, 9], the Riccati expansion method with constant coefficients [11, 12] or variable coefficients [13], etc. But there were also a few papers considering the doubly-periodic solutions which were expressed by using the Jacobi elliptic functions, Weierstrass elliptic function, the Theta functions, etc (see [14–19] and therein). Some methods were presented to seek the doubly-periodic solutions, such as the Jacobi elliptic function expansion method [14], the extended Jacobi elliptic function expansion method [15–17] and the algebraic method [20], etc. When the modulus $m \to 1$ or 0, the Jacobi elliptic functions degenerate as soliton solutions and trigonometric function solutions. Therefore, seeking the Jacobi elliptic function solutions of nonlinear wave equations is significant.

In order to seek more types of the Jacobi elliptic function solutions, in this paper, we would like to develop a transformation from the sinh-Gordon equation [1] which reveals a relationship between differential nonlinear wave equations. The transformation and sinh-Gordon equation are used to construct Jacobi elliptic function solutions of nonlinear wave equations. Under the transformation $u(x,t) = u(\xi)$, $\xi = k(x - \lambda t)$, the famous sinh-Gordon equation

$$\frac{\partial^2 \phi}{\partial x \partial t} = \alpha \sinh \phi \tag{1}$$

which appears in many branches of nonlinear science [1], where α is a constant, reduces to an ordinary differential equation

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\xi^2} = -\frac{\alpha}{k\lambda} \sinh \phi \tag{2}$$

where k and λ are the wave number and wave speed, respectively. Integrating (2) once yields

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{2}\phi\right)^2 = -\frac{\alpha}{k\lambda}\sinh^2\left(\frac{1}{2}\phi\right) + c \tag{3}$$

with integration constant c. If we set $c = 0$, $-\frac{\alpha}{k\lambda} = 1$, $\frac{1}{2}\phi = w$, then (3) becomes

$$\frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi} = \sinh w(\xi). \tag{4}$$

By using the solution of the equation (4), one can seek soliton solutions of nonlinear equations.

In this paper we would like to consider the case $c \neq 0$. In order to use (3) conveniently, we set $\phi = 2w$, $-\frac{\alpha}{k\lambda} = 1$, thus (3) reduces to

$$\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2 = \sinh^2 w + c \qquad \text{or} \qquad \frac{\mathrm{d}w}{\mathrm{d}\xi} = \sqrt{\sinh^2 w + c} \tag{5}$$

which is useful in the following method, where c is a constant of integration.

If we take $c = 1 - m^2$, where m $(0 < m < 1)$ is the modulus of the Jacobi elliptic functions [18], then we know that (5) with $c = 1 - m^2$ has the general solution

$$\sinh[w(\xi)] = \mathrm{cs}(\xi; m) \tag{6a}$$

or

$$\cosh[w(\xi)] = \mathrm{ns}(\xi; m) \tag{6b}$$

which are Jacobi elliptic functions and have the properties

$$\frac{\mathrm{d}\,\mathrm{cs}(\xi;m)}{\mathrm{d}\xi} = -\mathrm{ns}(\xi;m)\mathrm{ds}(\xi;m) \qquad \frac{\mathrm{d}\,\mathrm{ns}(\xi;m)}{\mathrm{d}\xi} = -\mathrm{cs}(\xi;m)\,\mathrm{ds}(\xi;m)$$
$$\mathrm{ns}^2(\xi;m) = 1 + \mathrm{cs}^2(\xi;m). \tag{7}$$

In what follows we would like to use the solution (6) of (5) to construct Jacobi elliptic function solutions of nonlinear wave equations.

The rest of this paper is arranged as follows. In section 2 we give the computational steps of method I, which is used for nonlinear ODEs with constant coefficients and nonlinear PDEs that can reduce to nonlinear ODEs with constant coefficients, and method II, which is used for nonlinear ODEs and nonlinear PDEs. In section 3 we use method I to construct the doubly-periodic solutions of some complex nonlinear wave equations in (1+1)-dimensional and (2+1)-dimensional spaces, such as the combined KdV-mKdV equation, (2+1)-dimensional coupled Davey–Stewartson equation, the new integrable Davey–Stewartson-type equation, the modified Boussinesq equation, (2+1)-dimensional mKP equation and the (2+1)-dimensional generalized KdV equation. Finally, we give some conclusions in section 4.

2. The sinh-Gordon equation expansion method and its algorithm

2.1. Method I—seeking the travelling wave type of Jacobi elliptic function solutions

For a given nonlinear partial differential equation, say, in two variables x, t

$$F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0 \tag{8}$$

we seek its travelling wave solution, if available, in the form $u(x, t) = u(\xi)$, $\xi = k(x - \lambda t)$. By using the new variable $w = w(\xi)$, we assume that (8) has the solution in the form

$$u(\xi) = u(w(\xi)) = A_0 + \sum_{i=1}^{n} \cosh^{i-1} w[A_i \sinh w + B_i \cosh w] \tag{9}$$

where A_i $(i = 0, 1, \ldots, n)$, B_j $(j = 1, 2, \ldots, n)$ are constants to be determined later and $w = w(\xi)$ satisfies (5).

According to (7) and (9), we define a polynomial degree function as $D(u(w)) = n$, thus we have

$$D\left(u^p(w)\left(\frac{\mathrm{d}^s u(w)}{\mathrm{d}\xi^s}\right)^q\right) = np + q(n + s). \tag{10}$$

Therefore, we can determine the parameter n by balancing the highest order derivative term with nonlinear terms in (8).

The method is summed up as the following steps:

Step 1. Reduce the given nonlinear equation to an ODE by using the travelling wave transformation $u(x, t) = u(\xi)$, $\xi = k(x - \lambda t)$.

Step 2. Determine the parameter n in (9) by balancing the highest order derivative terms and nonlinear terms and thus give the formal solution (9). (Remark: If n is not a positive integer, then we first make the transformation $u = v^n$, and then perform the second step again.)

Step 3. Substitute (9) with the known n along with (5) into the obtained ODE and obtain a hyperbolic polynomial for $w'^s \sinh^i w \cosh^j w$ $(i = 0, 1; s = 0, 1; j = 0, 1, 2, \ldots)$.

Step 4. Set to zero the coefficients of $w'^s \sinh^i w \cosh^j w$ $(i = 0, 1; s = 0, 1; j = 0, 1, 2, \ldots)$ to get a set of algebraic equations with respect to the unknowns k, λ, A_j $(j = 0, 1, \ldots, n)$ and B_j $(j = 1, 2, \ldots, n)$.

Step 5. Solve the set of algebraic equations, which may not be consistent, and finally derive the doubly-periodic solutions of the given nonlinear equations by using the $u(x, t) = u(\xi)$, $\xi = k(x - \lambda t)$ and the known solution (6).

Remark 1. This method is an indirect method which is used to find Jacobi elliptic function solutions of equations by using a transformation (9) and the target equation (5) with $c = 1-m^2$

whose solutions are known. Based on the symbolic computation, the procedure can be carried out by computer.

Remark 2. Because when $m \to 1, \mathrm{cs}(\xi; \mathrm{m}) \to \mathrm{csch}\,\xi$ and $\mathrm{ns}(\xi; \mathrm{m}) \to \coth\xi$; while $m \to 0, \mathrm{cs}(\xi; \mathrm{m}) \to \cot\xi$ and $\mathrm{ns}(\xi; \mathrm{m}) \to \csc\xi$, thus it is easy to see that the method is used to obtain both soliton solutions and Jacobi elliptic function solutions.

2.2. Method II—seeking the non-travelling wave type of Jacobi elliptic function solutions

We know that method I is only used for these nonlinear ODEs with constant coefficients or nonlinear partial differential equations that can be reduced to be the corresponding ODEs with constant coefficients by using some transformations, otherwise method I will not work. In order to overcome the disadvantage of method I, we change it into a general form as follows.

If we set $\xi \to \psi(x, t)$, then (5) becomes

$$\left(\frac{\mathrm{d}w(\psi)}{\mathrm{d}\psi}\right)^2 = \sinh^2 w(\psi) + c \qquad \text{or} \qquad \frac{\mathrm{d}w(\psi)}{\mathrm{d}\,\psi} = \sqrt{\sinh^2 w(\psi) + c} \tag{11}$$

where c is a constant of integration and $\psi(x, t)$ is an unknown function of x, t.

If we set $c = 1 - m^2$ $(0 < m < 1)$, then by solving (11), we know that it has the solution

$$\sinh[w(\psi)] = \mathrm{cs}[\psi(x, t); m] \tag{12a}$$

$$\cosh[w(\psi)] = \mathrm{ns}[\psi(x, t); m]. \tag{12b}$$

For the given nonlinear partial differential equation (8), we do not need to first make the travelling wave transformation to reduce (8) to an ODE with constant coefficients. We can directly assume that (8) has the generalized formal solution

$$u(x, t) = A_0(x, t) + \sum_{i=1}^{n} \cosh^{i-1} w(\psi)[A_i(x, t) \sinh w(\psi) + B_i(x, t) \cosh w(\psi)] \tag{13}$$

where the $w = w(\psi)$ satisfies (11) with $c = 1 - m^2$, and $A_i(x, t)$, $B_j(x, t)$ and $\psi(x, t)$ are functions to be determined later. Similar to the steps mentioned in method I, substituting (13) with (11) into (8) yields a set of differential equations w.r.t. A_i, B_i and ψ. By solving the set of differential equations, if available, and using (12), we can obtain more Jacobi elliptic function solutions. When the modulus $m \to 1$ or 0, we may obtain soliton-like solutions and more types of singly-periodic solutions.

Remark 3. If we take $\psi(x, t)$ to be of the form $\psi(x, t) = \xi = k(x - \lambda t)$ $(k, \lambda$ constants), then method II reduces to method I. But if we can obtain the case that the function $\psi(x, t)$ is not of the linearly combined form of x and t, then we will have new Jacobi elliptic function solutions of (8).

Remark 4. We know that method I transforms (8) into a system of nonlinear algebraic equations (SNAEs) with respect to unknown variables. But method II transforms (8) into a system of nonlinear partial differential equations (SNPDEs) with respect to unknown variables. Generally speaking, solving the SNPDEs is more difficult than the SNAEs. Thus, method II is more complicated than method I.

Recently, we have found new Jacobi elliptic function solutions of some simple nonlinear equations [21]. In what follows we would like to apply method I to some more complicated nonlinear equations, such as the combined KdV-mKdV equation, (2+1)-dimensional coupled Davey–Stewartson equation and (2+1)-dimensional generalized KdV equation, etc. As a consequence, some new Jacobi elliptic function solutions are obtained.

3. Some examples to illustrate method I and their solutions

In this section we illustrate method I using some nonlinear wave equations.

Example 3.1. The KdV-mKdV equation [1]

$$u_t + (\alpha + \beta u)uu_x + u_{xxx} = 0. \tag{14}$$

According to step 1, under the travelling wave transformation $u(x,t) = u(\xi)$, $\xi = k(x - \lambda t)$, (7) reduces to

$$C - \lambda u + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + k^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = 0 \tag{15}$$

where C is the integration constant. Fan [20] gave some exact solutions. In what follows we will give other types of Jacobi elliptic function solutions.

According to step 2, we assume that it has the solution

$$u(\xi) = A_0 + A_1 \sinh w(\xi) + B_1 \cosh w(\xi) \tag{16}$$

and w satisfying (4), $\xi = k(x - \lambda t)$.

With the aid of Maple, substituting (16) into (15) along with (5), we have the polynomial of $w'^s \sinh^i w \cosh^j w$. Setting their coefficients to zero yields a set of algebraic equations

$$\begin{aligned}
&1/3\beta B_1^3 + 2B_1k^2 + \beta A_1^2 B_1 k = 0\\
&\beta B_1^2 A_1 + 1/3\beta A_1^3 + 2A_1k^2 - 0\\
&\beta A_0 B_1^2 + 1/2\alpha A_1^2 + 1/2\alpha B_1^2 + \beta A_0 A_1^2 = 0\\
&A_1k^2c - A_1\lambda + \alpha A_0 A_1 + \beta A_0^2 A_1 - 1/3\beta A_1^3 - A_1k^2 = 0\\
&-2B_1k^2 - B_1\lambda + \alpha A_0 B_1 - \beta A_1^2 B_1 + B_1k^2c + \beta A_0^2 B_1 = 0\\
&\alpha A_1 B_1 + 2\beta A_0 A_1 B_1 = 0\\
&-1/2\alpha A_1^2 - \lambda A_0 + C - \beta A_0 A_1^2 + 1/2\alpha A_0^2 + 1/3\beta A_0^3.
\end{aligned} \tag{17}$$

From the above we have

$$A_0 = -\frac{\alpha}{2\beta} \qquad A_1 = 0 \qquad B_1 = \pm\sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(1+m^2)}} \qquad k = \pm\sqrt{-\frac{4\lambda\beta + \alpha^2}{4\beta(1+m^2)}} \tag{18}$$

$$A_0 = -\frac{\alpha}{2\beta} \qquad B_1 = 0 \qquad A_1 = \pm\sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(2-m^2)}} \qquad k = \pm\sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta(2-m^2)}} \tag{19}$$

$$A_0 = -\frac{\alpha}{2\beta} \qquad A_1^2 = B_1^2 A_1 = \pm\sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2(2m^2-1)}} \qquad k = \pm\sqrt{-\frac{4\lambda\beta + \alpha^2}{2\beta(1-2m^2)}}. \tag{20}$$

Therefore, we have three new Jacobi elliptic function solutions

$$u_1 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(1+m^2)}}\,\mathrm{cs}\left(\sqrt{-\frac{4\lambda\beta + \alpha^2}{4\beta(1+m^2)}}\xi\right) \tag{21}$$

$$u_2 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2(2-m^2)}}\,\mathrm{ns}\left(\sqrt{\frac{4\lambda\beta + \alpha^2}{4\beta(2-m^2)}}\xi\right) \tag{22}$$

$$u_3 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2(2m^2-1)}}[\mathrm{cs}(k\xi) \pm \mathrm{ns}(k\xi)] \qquad k = \sqrt{-\frac{4\lambda\beta+\alpha^2}{2\beta(1-2m^2)}}. \tag{23}$$

In particular

(1) when the modulus $m \to 1$, we have the soliton solutions from (21)–(23):

$$u_4 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}}\operatorname{csch}\left(\sqrt{-\frac{4\lambda\beta+\alpha^2}{8\beta}}\xi\right)$$

$$u_5 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2}}\coth\left(\sqrt{\frac{4\lambda\beta+\alpha^2}{4\beta}}\xi\right)$$

$$u_6 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}}[\operatorname{csch}(k\xi) \pm \coth(k\xi)] \qquad k = \sqrt{\frac{4\lambda\beta+\alpha^2}{2\beta}}$$

which are singular soliton solutions that imply that for the certain time $t = t_0$, these solutions blow up at the point $x = x_0$.

(2) when the modulus $m \to 0$, we have the singly-periodic solutions from (21)–(23):

$$u_7 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6(4\lambda\beta + \alpha^2)}{4\beta^2}}\cot\left(\sqrt{-\frac{4\lambda\beta+\alpha^2}{4\beta}}\xi\right)$$

$$u_8 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6(4\lambda\beta + \alpha^2)}{8\beta^2}}\csc\left(\sqrt{\frac{4\lambda\beta+\alpha^2}{8\beta}}\xi\right)$$

$$u_9 = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{3(4\lambda\beta + \alpha^2)}{4\beta^2}}[\csc(k\xi) \pm \cot(k\xi)] \qquad k = \sqrt{-\frac{4\lambda\beta+\alpha^2}{2\beta}}.$$

Example 3.2. (2+1)-dimensional coupled Davey–Stewartson equation [1, 22]

$$\begin{aligned} &\mathrm{i}\,u_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv = 0 \\ &v_{xx} + v_{yy} + 2(|u|)^2)_{xx} = 0. \end{aligned} \tag{24}$$

Fan [20] gave three Jacobi elliptic function solutions. In what follows we will obtain other types of Jacobi elliptic function solutions. We first introduce the transformations

$$u(x,t) = \exp(\mathrm{i}\eta)u(\xi) \qquad v(x,t) = v(\xi) \qquad \eta = \alpha x + \beta y + \gamma t \qquad \xi = k(x + py - \lambda t) \tag{25}$$

where $\alpha, \beta, \gamma, k, p, \lambda$ are constants to be determined later.

Substituting (25) into (24) we have

$$\begin{aligned} &k^2(1-p^2)u'' + (-\gamma - \alpha^2 + \beta^2)u - 2u^3 - 2uv = 0 \\ &(1+p^2)v'' + 2(u^2)'' = 0 \\ &\lambda = 2(\alpha - \beta p). \end{aligned} \tag{26}$$

Assume that (26) has the solutions, by using method I,

$$\begin{aligned} &u(\xi) = A_0 + A_1\sinh w + B_1\cosh w \\ &v(\xi) = a_0 + a_1\sinh w + b_1\cosh w + a_2\sinh w\cosh w + b_2\cosh^2 w \end{aligned} \tag{27}$$

where $A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2$ are constants to be determined later. According to the steps 3–5 in method I, we have the three new Jacobi elliptic function solutions:

$$u_1 = k\sqrt{1+p^2}\exp(\mathrm{i}\eta)\mathrm{cs}\xi \qquad v_1 = -2k^2\mathrm{cs}^2\xi + \frac{C}{1+p^2} \tag{28}$$

where $\eta = \alpha x + \beta y + \gamma t$, $\xi = k(x + py - 2(\alpha - p\beta)t)$, $k = \sqrt{\frac{2C/(1+p^2)+\gamma+\alpha^2-\beta^2}{(2-m^2)(1-p^2)}}$ and C is an arbitrary constant.

$$u_2 = k\sqrt{1+p^2}\exp(\mathrm{i}\eta)\mathrm{ns}\xi \qquad v_2 = -2k^2\mathrm{ns}^2\xi + \frac{C}{1+p^2} \tag{29}$$

where $\eta = \alpha x + \beta y + \gamma t$, $\xi = k(x + py - 2(\alpha - p\beta)t)$, $k = \sqrt{\frac{2C/(1+p^2)+\gamma+\alpha^2-\beta^2}{(1+m^2)(p^2-1)}}$ and C is an arbitrary constant.

$$u_3 = \frac{1}{2}k\sqrt{1+p^2}\exp(\mathrm{i}\eta)[\mathrm{cs}\xi \pm \mathrm{ns}\xi] \qquad v_3 = -k^2[\mathrm{cs}^2\xi \pm \mathrm{cs}\xi\,\mathrm{ns}\xi] - \frac{1}{2}k^2 + \frac{C}{1+p^2} \tag{30}$$

where $\eta = \alpha x + \beta y + \gamma t$, $\xi = k(x + py - 2(\alpha - p\beta)t)$, $k = \sqrt{\frac{2(2C/(q+p^2)+\gamma+\alpha^2-\beta^2)}{(1-2m^2)(1-p^2)}}$ and C is an arbitrary constant.

Similar to example 1, when the modulus $m \to 1$ or 0, we can also get the soliton solutions and singly-periodic solutions. We omit them here.

Example 3.3. The new integrable Davey–Stewartson-type equation [23]

$$\begin{aligned} &\mathrm{i}\Psi_\tau + L_1\Psi + \Psi\Phi + \Psi\chi = 0 \\ &L_2\chi = L_3|\Psi|^2 \\ &\Phi_\xi = \chi_\eta + \mu(|\Psi|^2)_\eta \qquad \mu = \mp 1 \end{aligned} \tag{31}$$

where the linear differential operators are given by

$$L_1 = \frac{b^2 - a^2}{4}\frac{\partial^2}{\partial\xi^2} - a\frac{\partial^2}{\partial\xi\partial\eta} - \frac{\partial^2}{\partial\eta^2} \tag{32a}$$

$$L_2 = \frac{b^2 + a^2}{4}\frac{\partial^2}{\partial\xi^2} + a\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2} \tag{32b}$$

$$L_3 = \pm\frac{1}{4}\left(b^2 + a^2 + \frac{8b^2(a-1)}{(a-2)^2 - b^2}\right)\frac{\partial^2}{\partial\xi^2} \pm \left(a + \frac{2b^2}{(a-2)^2 - b^2}\right)\frac{\partial^2}{\partial\xi\partial\eta} \pm \frac{\partial^2}{\partial\eta^2} \tag{32c}$$

where a, b are real parameters, and $\Psi = \Psi(\xi, \eta, \tau)$ is complex while $\Phi = \Phi(\xi, \eta, \tau)$, $\chi = \chi(\xi, \eta, \tau)$ are real. This equation was presented first by Maccari [23] from the Konopelchenko–Dubrovsky (KD) equation [24] by using the reduction method.

We first introduce the transformations

$$\begin{aligned} &\Psi(\xi,\eta,\tau) = \Psi(X)\exp(\mathrm{i}Y) \qquad \Phi(\xi,\eta,\tau) = \Phi(X) \qquad \chi(\xi,\eta,\tau) = \chi(X) \\ &X = kZ = k(\xi + l\eta + \lambda\tau) \qquad Y = \alpha\xi + \beta\eta + \gamma\tau \end{aligned} \tag{33}$$

where $k, l, \lambda, \alpha, \beta, \gamma$ are constants to be determined later.

Substituting (33) into (31), we have

$$\begin{aligned} &k^2M_1\frac{\mathrm{d}^2\Psi(X)}{\mathrm{d}X^2} + M_0\Psi(X) + \Psi(X)\Phi(X) + \Psi(X)\chi(X) = 0 \\ &k^2M_2\frac{\mathrm{d}^2\chi(X)}{\mathrm{d}X^2} = k^2M_3\frac{\mathrm{d}^2\Psi^2(X)}{\mathrm{d}X^2} \\ &k\frac{\mathrm{d}\Phi(X)}{\mathrm{d}X} = kl\frac{\mathrm{d}\chi(X)}{\mathrm{d}X} + \mu kl\frac{\mathrm{d}\Psi^2(X)}{\mathrm{d}X} \end{aligned} \tag{34a}$$

under the condition

$$\lambda = -\frac{\alpha}{4}(b^2 - a^2) + a(\beta + \alpha l) + 2l\beta \tag{34b}$$

where

$$\begin{aligned} M_0 &= -\gamma - \frac{1}{4}\alpha^2(b^2 - a^2) + a\alpha\beta + \beta^2 \\ M_1 &= \frac{b^2 - a^2}{4} - al - l^2 \qquad M_2 = \frac{b^2 + a^2}{4} + al + l^2 \\ M_3 &= \pm\frac{1}{4}\left(b^2 + a^2 + \frac{8b^2(a-1)}{(a-2)^2 - b^2}\right) \pm \left(a + \frac{2b^2}{(a-2)^2 - b^2}\right) l \pm l^2. \end{aligned} \tag{35}$$

According to method I, we assume that (34a) has the solution in the form

$$\begin{aligned} \Psi(X) &= A_0 + A_1 \sinh w + B_1 \cosh w \\ \Phi(\xi) &= a_0 + a_1 \sinh w + b_1 \cosh w + a_2 \sinh w \cosh w + b_2 \cosh^2 w \\ \chi(\xi) &= e_0 + e_1 \sinh w + f_1 \cosh w + e_2 \sinh w \cosh w + f_2 \cosh^2 w \end{aligned} \tag{36}$$

where $A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2, e_0, e_1, e_2, f_1, f_2$ are constants to be determined later. According to the steps 3–5 in method I, we have the three new Jacobi elliptic function solutions:

$$\Psi_1 = \sqrt{-\frac{2(M_0 + c_1 + lc_1 + c_2)M_2}{(1+m^2)(M_3 + lM_3 + \mu l M_2)}}\,\mathrm{ns}\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(1+m^2)M_1}}Z\right]\exp(\mathrm{i}Y) \tag{37a}$$

$$\chi_1 = -\frac{2M_3(M_0 + c_1 + lc_1 + c_2)}{(1+m^2)(M_2 + lM_2 + \mu l M_3)}\,\mathrm{ns}^2\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(1+m^2)M_1}}Z\right] + c_1 \tag{37b}$$

$$\Phi_1 = -\frac{2(M_0 + c_1 + lc_1 + c_2)M_2}{(1+m^2)(M_3 + lM_3 + \mu l M_2)}\left(\frac{lM_3}{M_2} + \mu l\right)\mathrm{ns}^2\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(1+m^2)M_1}}Z\right] + lc_1 + c_2 \tag{37c}$$

$$\Psi_2 = \sqrt{-\frac{2(M_0 + c_1 + lc_1 + c_2)M_2}{(m^2-2)(M_3 + lM_3 + \mu l M_2)}}\,\mathrm{cs}\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(m^2-2)M_1}}Z\right]\exp(\mathrm{i}Y) \tag{38a}$$

$$\chi_2 = -\frac{2M_3(M_0 + c_1 + lc_1 + c_2)}{(m^2-2)(M_2 + lM_2 + \mu l M_3)}\,\mathrm{cs}^2\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(m^2-2)M_1}}Z\right] + c_1 \tag{38b}$$

$$\Phi_2 = -\frac{2(M_0 + c_1 + lc_1 + c_2)M_2}{(m^2-2)(M_3 + lM_3 + \mu l M_2)}\left(\frac{lM_3}{M_2} + \mu l\right)\mathrm{cs}^2\left[\sqrt{\frac{M_0 + c_1 + lc_1 + c_2}{(m^2-2)M_1}}Z\right] + lc_1 + c_2 \tag{38c}$$

$$\begin{aligned} \Psi_3 = \sqrt{-\frac{(M_0 + c_1 + lc_1 + c_2)M_2}{(2m^2-1)(M_3 + lM_3 + \mu l M_2)}}&\left\{\mathrm{ns}\left[\sqrt{\frac{2(M_0 + c_1 + lc_1 + c_2)}{(2m^2-1)M_1}}Z\right]\right. \\ &\left.\pm\,\mathrm{cs}\left[\sqrt{\frac{2(M_0 + c_1 + lc_1 + c_2)}{(2m^2-1)M_1}}Z\right]\right\}\exp(\mathrm{i}Y) \end{aligned} \tag{39a}$$

$$\chi_3 = -\frac{(M_0+c_1+lc_1+c_2)M_3}{(2m^2-1)(M_3+lM_3+\mu lM_2)}\left\{2\mathrm{cs}^2\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]+1 \pm 2\mathrm{ns}\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]\mathrm{cs}\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]\right\}+c_1 \tag{39b}$$

$$\Phi_3 = -\frac{(M_0+c_1+lc_1+c_2)M_2}{(2m^2-1)(M_3+lM_3+\mu lM_2)}\left(\frac{lM_3}{M_2}+\mu l\right)\left\{2\mathrm{cs}^2\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]+1 \pm 2\mathrm{ns}\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]\mathrm{cs}\left[\sqrt{\frac{2(M_0+c_1+lc_1+c_2)}{(2m^2-1)M_1}}Z\right]\right\}+lc_1+c_2 \tag{39c}$$

where c_1 and c_2 are constants.

Example 3.4. The modified Boussinesq equation [25]

$$\begin{aligned} P_t &= \left(Q-\frac{3}{2}k^2P^2\right)_x \\ Q_t &= -3k^2(P_{xx}-PQ+k^2P^3)_x \end{aligned} \tag{40}$$

where k is a constant.

Under the travelling wave transformation

$$P(x,t)=P(\xi) \qquad Q(x,t)=Q(\xi) \qquad \xi=\alpha(x+\lambda t) \tag{41}$$

(40) reduces to

$$\begin{aligned} \lambda\frac{\mathrm{d}P}{\mathrm{d}\xi} &= \frac{\mathrm{d}Q}{\mathrm{d}\xi}-\frac{3}{2}k^2\frac{\mathrm{d}P^2}{\mathrm{d}\xi} \\ \lambda\frac{\mathrm{d}Q}{\mathrm{d}\xi} &= -3k^2\left(\alpha^2\frac{\mathrm{d}^3P}{\mathrm{d}\xi^3}-\frac{\mathrm{d}(PQ)}{\mathrm{d}\xi}+k^2\frac{\mathrm{d}P^3}{\mathrm{d}\xi}\right). \end{aligned} \tag{42}$$

Assume that (42) has the solutions, by using method I,

$$\begin{aligned} P(\xi) &= A_0+A_1\sinh w+B_1\cosh w \\ Q(\xi) &= a_0+a_1\sinh w+b_1\cosh w+a_2\sinh w\cosh w+b_2\cosh^2 w \end{aligned} \tag{43}$$

where $A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2$ are constants to be determined later. According to the steps 3–5 in method I, we have the three new Jacobi elliptic function solutions:

$$P_1 = \sqrt{\frac{2(\lambda^2-2k^2c_1)}{k^4(1+m^2)}}\mathrm{ns}\left[\sqrt{\frac{\lambda^2-2k^2c_1}{2k^2(1+m^2)}}(x+\lambda t)\right]-\frac{\lambda}{3k^2} \tag{44a}$$

$$Q_1 = \frac{3(\lambda^2-2k^2c_1)}{k^2(1+m^2)}\mathrm{ns}^2\left[\sqrt{\frac{\lambda^2-2k^2c_1}{2k^2(1+m^2)}}(x+\lambda t)\right]-\frac{\lambda^2}{6k^2}+c_1 \tag{44b}$$

$$P_2 = \sqrt{\frac{2(\lambda^2-2k^2c_1)}{k^4(m^2-2)}}\mathrm{cs}\left[\sqrt{\frac{\lambda^2-2k^2c_1}{2k^2(m^2-2)}}(x+\lambda t)\right]-\frac{\lambda}{3k^2} \tag{45a}$$

$$Q_2 = \frac{3(\lambda^2 - 2k^2c_1)}{k^2(m^2-2)}\mathrm{cs}^2\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{2k^2(m^2-2)}}(x+\lambda t)\right] - \frac{\lambda^2}{6k^2} + c_1 \tag{45b}$$

$$P_3 = \sqrt{\frac{\lambda^2 - 2k^2c_1}{k^4(2m^2-1)}}\left\{\mathrm{ns}\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{k^2(2m^2-1)}}(x+\lambda t)\right] \pm \mathrm{cs}\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{k^2(2m^2-1)}}(x+\lambda t)\right]\right\} - \frac{\lambda}{3k^2} \tag{46a}$$

$$Q_3 = \frac{3(\lambda^2 - 2k^2c_1)}{2k^2(2m^2-1)}\left\{2\mathrm{cs}^2\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{k^2(2m^2-1)}}(x+\lambda t)\right] \pm 2\mathrm{ns}\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{k^2(2m^2-1)}}(x+\lambda t)\right]\right.$$
$$\left.\times\,\mathrm{cs}\left[\sqrt{\frac{\lambda^2 - 2k^2c_1}{k^2(2m^2-1)}}(x+\lambda t)\right] + 1\right\} - \frac{\lambda^2}{6k^2} + c_1 \tag{46b}$$

where c_1 is a constant.

Example 3.5. (2+1)-dimensional mKP equation [24, 26]

$$q_t + \tfrac{1}{8}\left(q_{xxx} - 6q^2q_x + 6q_x\partial_x^{-1}q_y + 3\partial_x^{-1}q_{yy}\right) = 0. \tag{47}$$

Dai [26] used a direct method to decompose (47) into two (1+1)-dimensional soliton equations. By using the Darbooux transformation of the two (1+1)-dimensional soliton equations, some soliton solutions of (47) were obtained. In what follows we consider its Jacobi elliptic function solutions.

According to method I, we can obtain three new Jacobi elliptic function solutions of (47):

$$q_1 = \sqrt{\frac{9/2l^2 + 8\lambda}{1+m^2}}\mathrm{ns}\left[\sqrt{\frac{9/2l^2 + 8\lambda}{1+m^2}}(x+ly+\lambda t)\right] + \frac{1}{2}l \tag{48}$$

$$q_2 = \sqrt{\frac{9/2l^2 + 8\lambda}{m^2-2}}\mathrm{cs}\left[\sqrt{\frac{9/2l^2 + 8\lambda}{m^2-2}}(x+ly+\lambda t)\right] + \frac{1}{2}l \tag{49}$$

$$q_3 = \sqrt{\frac{9/2l^2 + 8\lambda}{2(2m^2-1}}\left\{\mathrm{ns}\left[\sqrt{\frac{9l^2 + 16\lambda}{2m^2-1}}(x+ly+\lambda t)\right] \pm \mathrm{cs}\left[\sqrt{\frac{9l^2 + 16\lambda}{2m^2-1}}(x+ly+\lambda t)\right]\right\} + \frac{1}{2}l. \tag{50}$$

Example 3.6. The (2+1)-dimensional generalized KdV equation [27]

$$u_t + a_1u_{xxx} + a_2u_{yyy} + a_3u_x + a_4u_y - 3a_1\left(u\partial_y^{-1}u_x\right)_x - 3a_2\left(u\partial_x^{-1}u_y\right)_y = 0 \tag{51}$$

which is obtained by Boiti [27] forms the general equation, where a_i $(i = 1, 2, 3, 4)$ are constants. When $a_3 = a_4 = 0$, (22) reduces to the Nizhhnik–Novikov–Veselov equation [28].

According to the above-mentioned method I, we can obtain two new types of Jacobi elliptic function solutions

$$u_1 = 2k^2p\,\mathrm{cs}^2\xi - \frac{p(a_4p - \lambda + (4a_2k^2p^3 + 4a_1k^2)(2-m^2) + a_3)}{6(a_1 + a_2p^3)} \tag{52}$$

$$u_2 = k^2 p[\mathrm{cs}^2\xi \pm \mathrm{cs}\,\xi \mathrm{ns}\xi] + \frac{1}{2}k^2 p - \frac{p(a_4 p - \lambda + (2a_2k^2p^3 + 2a_1k^2)(1-2m^2) + a_3)}{6(a_1 + a_2p^3)}. \quad (53)$$

Similar to example 1, when the modulus $m \to 1$ or 0, we can also get the soliton solutions and singly-periodic solutions:

$$u_3 = 2k^2 p \,\mathrm{csch}^2\,\xi - \frac{p(a_4 p - \lambda + 4a_2k^2p^3 + 4a_1k^2 + a_3)}{6(a_1 + a_2p^3)} \quad (54)$$

$$u_4 = 2k^2 p \cot^2\xi - \frac{p(a_4 p - \lambda + 8a_2k^2p^3 + 8a_1k^2 + a_3)}{6(a_1 + a_2p^3)} \quad (55)$$

$$u_5 = k^2 p[\mathrm{csch}^2\,\xi \pm \mathrm{csch}\,\xi \coth\xi] + \frac{1}{2}k^2 p - \frac{p(a_4 p - \lambda - (2a_2k^2p^3 + 2a_1k^2) + a_3)}{6(a_1 + a_2p^3)} \quad (56)$$

$$u_6 = k^2 p[\cot^2\xi \pm \csc\xi \cot\xi] + \frac{1}{2}k^2 p - \frac{p(a_4 p - \lambda + (2a_2k^2p^3 + 2a_1k^2) + a_3)}{6(a_1 + a_2p^3)} \quad (57)$$

where $\xi = k(x + py - \lambda t)$.

4. Conclusions

In summary, based on the second-order sinh-Gordon equation, we have developed a new method (called the sinh-Gordon equation expansion method). Method I is used on the combined KdV-mKdV equation, (2+1)-dimensional coupled Davey–Stewartson equation, the new integrable Davey–Stewartson-type equation, a modified Boussinesq equation, (2+1)-dimensional mKP equation and the (2+1)-dimensional generalized KdV equation such that some new solutions are obtained. When the modulus $m \to 1$ or 0, the corresponding solitary waves and singly periodic solutions are also found. Therefore, it is easily seen that the method is simple and powerful and can be carried out by computer with the aid of symbolic computation. But we know that method I is only used on these nonlinear ODEs with constant coefficients and nonlinear PDEs that can reduce to nonlinear ODEs with constant coefficients. If method I does not work for some nonlinear differential equations, then we may use method II to solve them. When one applies method II to nonlinear differential equations, it is complex to solve. Sometimes we may not obtain more results by using method II than method I. Examples of applications of method II will be given in the future.

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